MAT1322

Lecture 19

27/03/2017

Tangent planes, linear approximations

Recall: partial derivatives

Compute f_{xyy} of $f(x,y) = 3x^4y^2 + \frac{x}{y^2}$

$f_{xyy} = ((f_x)_y)_y$ → start deriving for x, then y twice

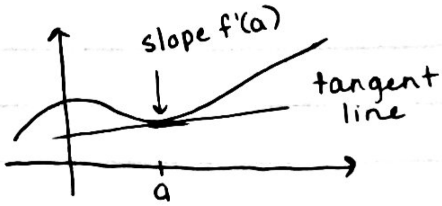
$f_x = 12x^3y^2 + \frac{1}{y^2}$

$f_{xy} = 24x^3y - \frac{2}{y^3}$

$\underline{f_{xyy} = 24x^3 + \frac{6}{y^4}}$

TANGENT PLANES (14.4)

Recall in 2D: (Calc I)



- tangent line eq'n at point a

The partial derivatives are the slopes along (parallel) to the x-axis and y-axis.

Tangent line equation:

f has continuous first partial derivatives. An equation for the tangent plane to $z=f(x,y)$ at point (x_0, y_0, z_0) ($z_0 = f(x_0, y_0)$) ← given

$$z - z_0 = f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$$

Ex $f(x,y) = 2x^2+y^2$, tangent plane at $(\underset{x_0}{1}, \underset{y_0}{1})$

$f(1,1) = 2(1)^2+(1)^2$ → computing z_0

$= 3$ ∴ at point $(1,1,3)$

Compute f_x and f_y:

$f_x = 4x \quad f_x(x_0,y_0) = f_x(1,1) = 4 \cdot 1 = \underline{4}$

$f_y = 2y \quad f_y(1,1) = \underline{2}$

$z - 3 = 4(x-1) + 2(y-1)$

$z = 4x - 4 + 2y - 2 + 3$

$\boxed{z = 4x + 2y - 3}$

LINEAR APPROXIMATIONS

As in 2D, we can use the tangent line equation to get a linear approximation.

<u>Recall</u>: Taylor series

$$T_{f,a}(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

<u>Linear approx</u>:

$$L_{f,a}(x) = f(a) + f'(a)(x-a)$$

(first 2 terms of Taylor series!)

In 3D, take the tangent plane equation and transform it into a function:

$$z - z_0 = f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$$

$$L_{f,(x_0,y_0)}(x,y) = f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0) \underline{+ z_0} \quad = z$$

↳ as in 2D, this is only accurate near the original point

DIFFERENTIABILITY

So far, we discussed when partial derivatives exist (2D limits)

<u>Def</u> $z=f(x,y)$ is differentiable at (a,b) if Δz can be expressed in the form

$$\Delta z = f_x(a,b)\Delta x + f_y(a,b)\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$$

$$\varepsilon_1, \varepsilon_2 \to 0 \quad \text{as} \quad (\Delta x, \Delta y) \to (0,0)$$

We use the tangent plane and the linear approx. via it to define differentiability.

<u>Theorem</u> If the partial derivatives f_x, f_y exist near (a,b) and are continuous at (a,b), then f is differentiable at (a,b).

TOTAL DIFFERENTIALS

A total differential of a function $f(x,y)$ is defined as:

$$d_z = f_x(x,y)\underbrace{dx}_{} + f_y(x,y)\underbrace{dy}_{}$$

abstract symbols!

Ex $f(x,y) = x^2 + 3xy - y^2$

$f_x(x,y) = 2x + 3y$ $\Rightarrow$ $dz = (2x+3y)dx + (3x-2y)dy$

$f_y(x,y) = 3x - 2y$

* related to tangent plane equation: $dz = z - z_0, \ldots$

↳ use it to get the tangent plane at (a,b).

Ex at $(1,1)$

so: $\Delta x = x - 1$, $\Delta y = y - 1 \ldots$

CHAIN RULE (14.5)

recall: ex $f(x) = e^{3x^2}$, $f'(x) = e^{3x^2} \cdot 6x$

$f(x) = g(h(x))$, $g(x) = e^x$, $h(x) = 3x^2$

↑ composition of functions

chain rule in 2D: $f'(x) = g'(h(x))\, h'(x)$

In 3D:

Chain rule CASE 1: $z = f(x(t), y(t))$

$x(t)$, $y(t)$ both differentiable (in sense of calc 1)

$$\frac{dz}{dt} = \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt}$$

Ex $z = x^2y + 3xy^4$, where $x(t) = \sin(2t)$

$y(t) = \cos(t)$

(so hidden: $f(x,y) = x^2y + 3xy^4$)

naively: substitute $x(t), y(t)$ in z:

$$z = \underbrace{\sin^2(2t)\cos(t) + 3\sin(2t)\cos^4(t)}_{}$$

deriving this for t is long, so use chain rule instead →

formula: $\frac{dz}{dt} = \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt}$

$f(x) = x^2y + 3xy^4$

$x(t) = \sin(2t)$
$y(t) = \cos(t)$

formula needs: $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{dx}{dt}, \frac{dy}{dt}$

$f_x = \frac{\partial f}{\partial x} = \underline{2xy + 3y^4}$

$\frac{dx}{dt} = (\sin(2t))' = \underline{\cos(2t) \cdot 2}$ ← chain rule fr. calc I

$f_y = \frac{\partial f}{\partial y} = \underline{x^2 + 12xy^3}$

$\frac{dy}{dt} = (\cos(t))' = \underline{-\sin(t)}$

put in formula:

$$\boxed{\frac{dz}{dt} = (2xy + 3y^4)(2\cos(2t)) + (x^2 + 12xy^3)(-\sin(t))}$$

not final answer! need to substitute

can now substitute for x & y: ($x = \sin(2t)$, $y = \cos(t)$)

☆ $$\boxed{\frac{dz}{dt} = 2\sin(2t)\cos(t) + 3(\cos^4(t))(2\cos(2t)) + (\sin^2(2t) + 12\sin(2t)\cos^3(t))(-\sin(t))}$$

$\frac{dz}{dt}$ NEEDS to be an expression only in t.